MAT 545 Programming Project

Diego G. Avalos

Winter 2017

Abstract

We study the delayed logistic equation $N' = rN(1 - N(t - \tau)/K)$ and seek to determine the stability of the fixed point $N^* = K$. After applying convenient transformations we use Matlab to generate various solution plots and observe that a change of stability occurs when $r\tau$ is somewhere near 1.6. The problem becomes equivalent to studying the stability of our new equation $u' = -\lambda(u+1)u(x-1)$, where $\lambda = r\tau$, at the fixed point $u^* = 0$. Furthermore, we linearize our transformed equation to verify such change in stability exists and conclude that it occurs when the values of our parameter λ changes at $\pi/2$. The solutions to our initial equation are stable for $\lambda \in (0, \pi/2)$, eventually periodic for $\lambda = \pi/2$, and unstable for $\lambda \in (\pi/2, \infty)$.

Introduction

The continuous logistic equation was published in 1838 by Pierre Verhulst, and it is used for population models because it implements a population-limiting parameter K, called the carrying capacity, which makes it a more convenient and realistic equation to use than the simpler exponential growth model. The logistic equation is given by

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) \tag{1}$$

where $N = N(t)$ is the population at time t, $r > 0$ is the growth rate, and $K > 0$ is the carrying capacity.

However, if we intend to go a step further and fashion a yet more realistic population model, we may introduce a delay to equation (1). This is often necessary because it has been observed that in some population and other biological models the rate of change of the population does not only depend on the present time population data but also in the population data at some point of the past because the process of reproduction is not always instantaneous. In other words, the rate of change of a population sometimes depends on the number of individuals of a past generation.

We introduce the delay by making the per capita growth rate N'/N of equation (1) depend on the value $N(t-\tau)$ instead of $N(t)$, where $\tau > 0$ represents the delay. Hence, we modify equation (1) to the delayed logistic equation

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right) \tag{2}$$

where $\tau > 0$ is the delay. The equation above was proposed by G. Evelyn Hutchinson in 1948.

The objective of this project is to study the behavior of solutions of equation (2) and to explore the stability of the fixed point $N^* = K$ of the delayed differential equation.

Discussion and Results

To study the solutions of equation (2), we write it in dimensionless form by using the transformations $y(x) = N(t)/K$ and $x = t/\tau$. With this in mind, observe that

$$\frac{dy}{dx} = \frac{d}{dx}[N(t)/K] = \frac{d}{dt}[N(t)/K]\frac{dt}{dx} = \frac{d}{dt}[N(t)/K]\frac{d(x\tau)}{dx} = \frac{\tau}{K}\frac{dN(t)}{dt}. \tag{3}$$

Moreover, since $N(\tau x) = Ky(x)$ we have

$$N(t-\tau) = N(\tau x - \tau) = N[\tau(x-1)] = Ky(x-1). \tag{4}$$

Substituting (3) and (4) into (2) we obtain $\frac{K}{\tau}\frac{dy}{dx} = rKy(x)(1 - Ky(x-1)/K)$, or equivalently

$$\frac{dy}{dx} = \lambda y(1 - y(x-1)) \tag{5}$$

where $\lambda = r\tau > 0$. Note that our fixed point $N^* = K$ has been transformed to $y^* = 1$. From now on, we will consider the initial history $y = 1/2$ on $-1 \le x \le 0$ for the DDE in (5). Let us attempt to solve equation (5) by the method of steps. On the interval $[0,1]$ our DDE becomes the ODE

$$\frac{dy}{dx} = \frac{\lambda y}{2}$$

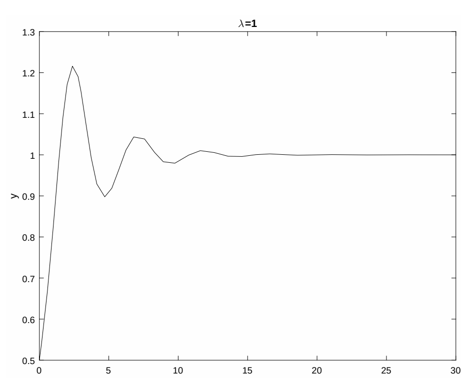
with initial condition $y(0) = 1/2$. Thus, our solution for $0 \le x \le 1$ is $y = \frac{1}{2}e^{\frac{\lambda}{2}x}$. On the interval $[1,2]$ we have the ODE

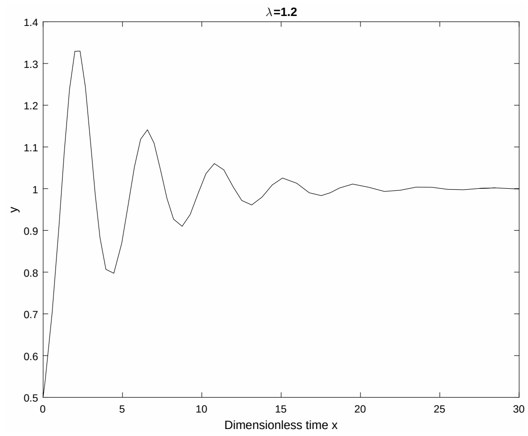
$$\frac{dy}{dx} = \lambda y[1 - \frac{1}{2}e^{\frac{\lambda}{2}(x-1)}]$$

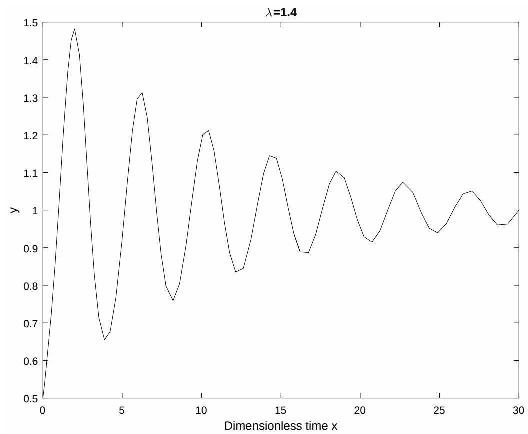
with initial condition $y(1) = \frac{1}{2}e^{\frac{\lambda}{2}}$. Hence, by separation of variables we obtain the solution

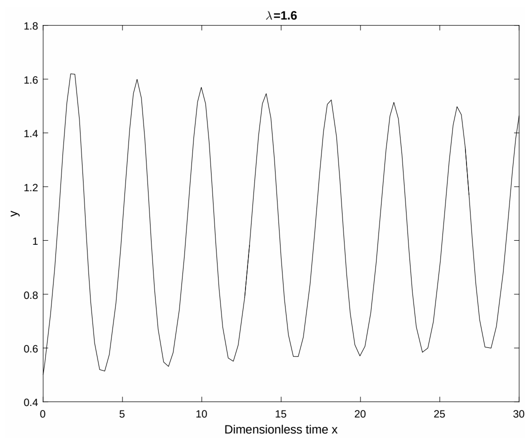
$$y = \frac{1}{2}e^{1-\frac{\lambda}{2}}\exp(\lambda x - e^{\frac{\lambda}{2}(x-1)})$$

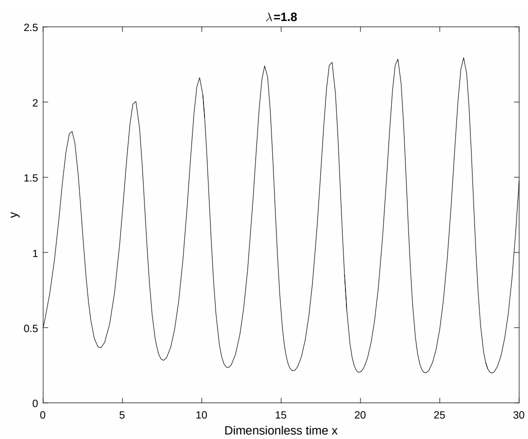
for $1 \le x \le 2$. Observe that the higher the intervals we attempt to solve our DDE exactly, the more complicated and overwhelming the method of steps becomes since we need to integrate the previous solution over x by separation of variables, which leads us to a series of nested exponential functions that need to be integrated. This is a difficult problem, so we shall study the behavior of the solutions of the DDE in (5) with the aid of Matlab. Consider the following solution plots of our DDE in equation (5) for various values of λ between 1 and 2.2.

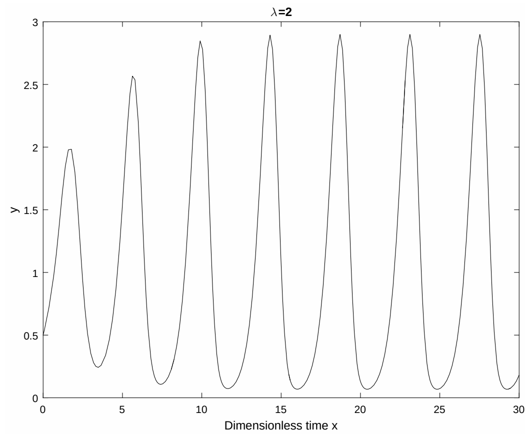


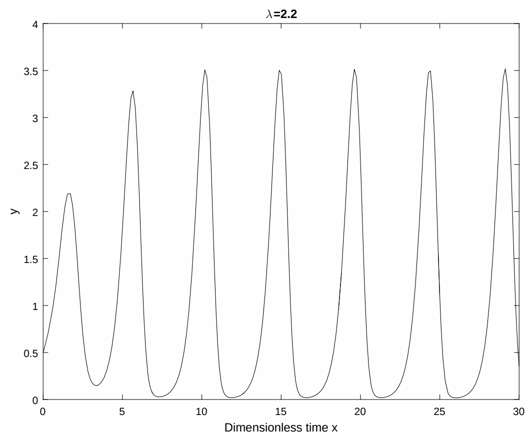

λ=1.2
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
y
0
5
10
15
20
25
30
Dimensionless time x


λ=1.4
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
y
0
5
10
15
20
25
30
Dimensionless time x


λ=1.6
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
y
0
5
10
15
20
25
30
Dimensionless time x


λ=1.8
2.5
2
1.5
1
0.5
0
y
0
5
10
15
20
25
30
Dimensionless time x





We now begin to study the stability of $y^* = 1$, a fixed point of (5). Observe that the plots suggest the stability of the solutions of equation (5) change at some value of λ, which we call λ_0, close to 1.6 from stable to unstable. We make yet another substitution $u = y - 1$ to help us linearize (5) around y^* in order to determine the bifurcation value λ_0. Our substitution yields

$$\frac{du}{dx} = -\lambda(u(x) + 1)u(x - 1) \tag{6}$$

but since we are creating perturbations near $y^* = 1$, we are close to $u^* = 0$. Thus $u(x) + 1 \approx 1$, so (6) becomes

$$\frac{du}{dx} = -\lambda u(x - 1). \tag{7}$$

Observe that (7) is the linearization of our DDE (6) around the fixed point $u^* = 0$; this will help us find the bifurcation value λ_0 of equation (5) at $y^* = 1$. To explore the stability of y^* we suppose the ODE in (7) has solutions of the form

$$u = e^{zx} \tag{8}$$

where $z \in \mathbb{C}$. Since y^* is stable when the real part of z is negative, we will let $z = a + bi$, where $a, b \in \mathbb{R}$ and seek for the λ-values where a changes from negative to positive. That is, first we substitute (8) in (7) to get $ze^{zx} = -\lambda e^{zx}e^{-z}$. This yields the characteristic equation

$$z = -\lambda e^{-z} \tag{9}$$

and in terms of its real and imaginary parts, (9) becomes

$$a + bi = -\lambda e^{-a}(\cos b - i \sin b). \tag{10}$$

From (10) we can extract the real and imaginary parts and obtain the following two equations

$$a = -\lambda e^{-a} \cos b \tag{11}$$

$$b = \lambda e^{-a} \sin b. \tag{12}$$

To find where a changes sign from negative to positive, we set $a = 0$ in the two equations above. This yields

$$0 = -\lambda \cos b \tag{13}$$

$$b = \lambda \sin b. \tag{14}$$

Since $\lambda > 0$, equation (13) yields $b_n = \pi/2 + n\pi$ for $n \in \mathbb{Z}$ as the solutions for b. These in turn make equation (14) become $\lambda_n = (-1)^n b_n$. Since some of the λ_n-values repeat for some n and they also are positive, it suffices to simply take the even nonnegative values of n into consideration. Also, we must only use the smallest solution for b since it is what dictates the behavior of the real and imaginary parts of z over the rest of the b values because it is the one that occurs first relative to λ, which is taken as a changing parameter. For instance, we only consider $b_0 = \pi/2$. That is, we are interested in the point $P_0 = (\lambda_0, a_0, b_0) = (\pi/2, 0, \pi/2)$. The following theorem will help us with our theory.

Theorem. *(A version of the implicit function theorem) Let $F(\lambda, a, b) = \begin{bmatrix} f(\lambda, a, b) \\ g(\lambda, a, b) \end{bmatrix}$ be a function such that both $f, g : \mathbb{R}^3 \to \mathbb{R}$ are continuously differentiable, and let $P_0 = (\lambda_0, a_0, b_0) \in \mathbb{R}^3$ be fixed such that $F(P_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Define the Jacobian matrix*

$$J_{a,b} = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \\ \frac{\partial g}{\partial a} & \frac{\partial g}{\partial b} \end{bmatrix}$$

and suppose that $J_{a,b}(P_0)$ is invertible. Then there exists a unique continuously differentiable implicit function $G(\lambda) = \begin{bmatrix} a(\lambda) \\ b(\lambda) \end{bmatrix}$ and open sets U, V such that $G : U \to V$ where $U \subseteq \mathbb{R}$ and $V \subseteq \mathbb{R}^2$ are open, $\lambda_0 \in U$, $(a_0, b_0) \in V$, and $F(\lambda, G(\lambda)) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all $\lambda \in U$. Moreover, we can obtain the derivatives of a and b in implicit form whenever $J_{a,b}$ is nonsingular from the equation

$$\begin{bmatrix} \frac{da}{d\lambda} \\ \frac{db}{d\lambda} \end{bmatrix} = -J_{a,b}^{-1}(\lambda, a, b) \begin{bmatrix} \frac{\partial f}{\partial \lambda} \\ \frac{\partial g}{\partial \lambda} \end{bmatrix}.$$

We will use our theorem to show that there exist continuous parametrized implicit functions $a(\lambda)$ and $b(\lambda)$ which satisfy equations (11) and (12) for all $\lambda > 0$. Define the functions

$$f(\lambda, a, b) = -\lambda e^{-a} \cos b - a$$

$$g(\lambda, a, b) = \lambda e^{-a} \sin b - b$$

and note that they are continuously differentiable everywhere including all positive values of λ. At $P_0 = (\pi/2, 0, \pi/2)$, we have $f(P_0) = g(P_0) = 0$. Furthermore, the Jacobian matrix $J_{a,b}$ is

$$J_{a,b} = \begin{bmatrix} \lambda e^{-a} \cos b - 1 & \lambda e^{-a} \sin b \\ -\lambda e^{-a} \sin b & \lambda e^{-a} \cos b - 1 \end{bmatrix}$$

and $\det(J_{a,b}) = (\lambda e^{-a}\cos b - 1)^2 + (\lambda e^{-a}\sin b)^2$ is only zero when $a = \log\lambda$. Hence, $J_{a,b}(P_0)$ is invertible, so by our theorem there exist unique continuously differentiable implicit functions $a(\lambda)$ and $b(\lambda)$ for $\lambda > 0$ such that $f(\lambda_0, a(\lambda_0), b(\lambda_0)) = g(\lambda_0, a(\lambda_0), b(\lambda_0)) = 0$, and satisfy equations (11) and (12) for all $\lambda > 0$. Furthermore, our theorem allows us to compute the implicit derivative

$$\frac{da}{d\lambda} = \frac{\lambda e^{-a} - \cos b}{e^a \det(J_{a,b})}. \tag{15}$$

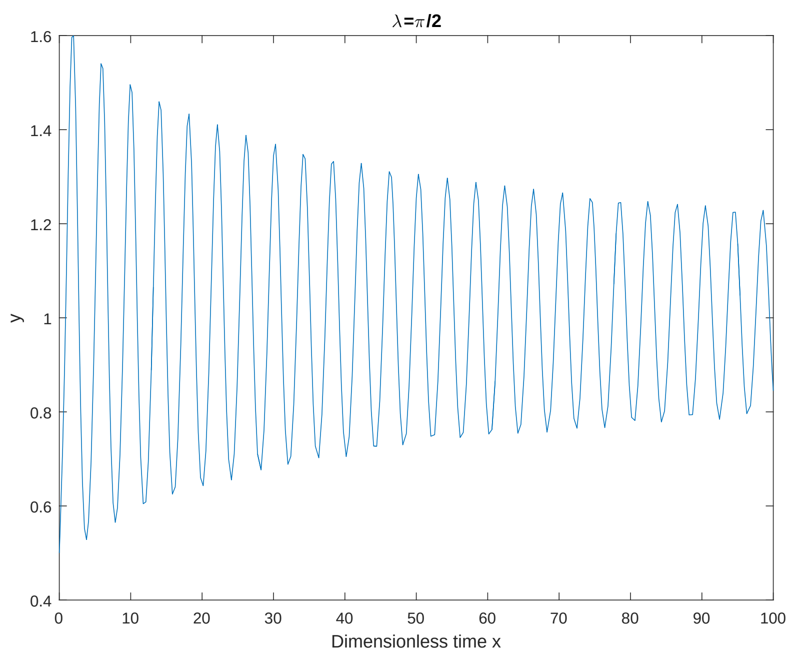
In the λ-a plane, $a(\lambda)$ passes through $(\lambda_0, a_0) = (\pi/2, 0)$ with a positive derivative $\frac{da}{d\lambda}\big|_{P_0} = \frac{\pi}{2}$. Since $a(\lambda)$ is unique and has a continuous derivative, its graph lies bellow the curve $a = \log\lambda$, that is, for all $\lambda > 0$ we have $a < \log\lambda$. It follows from the latter inequality that $\lambda e^{-a} > 1 \geq \cos b$, then $\lambda e^{-a} - \cos b > 0$ for all $\lambda > 0$, so the derivative in (15) is always positive. Therefore, $a(\lambda)$ is always increasing.
We may conclude the following from our analysis

1. If $\lambda \in (0, \pi/2)$, the real part of z in equation (8) is always negative, so the fixed point $y^* = 1$ of our DDE in (5) is stable.

2. If $\lambda \in (\pi/2, \infty)$, the real part of z in equation (8) is always positive, so the fixed point $y^* = 1$ of our DDE in (5) is unstable.

3. If $\lambda = \pi/2$, the real part of z in equation (8) is zero, so the solution eventually has periodic oscillations around the fixed point $y^* = 1$ of our DDE in (5).

The method to obtain our last conclusion is we consider the plot of the solution of (5) for $\lambda = 0$, which we show bellow.



Observe that around $y^* = 1$ the oscillations of the solution appear to eventually have a steady amplitude. In reality an extra decrease in amplitude occurs in every plot presented in this paper because of the damping effect created by both the `dde23` Matlab function and the fact that we neglected the nonlinear term in equation (6).

References

[1] Bertram, Richard. Delay Differential Equations.

[2] Rao, Milind M. and Preetish, K. L. Stability and Hopf Bifurcation Analysis of the Delay Logistic Equation.

[3] Ruan, Shigui. Delay Differential Equations in Single Species Dynamics.

[4] Wikipedia. Implicit Function Theorem.

Source Codes

This is the Matlab code which generates our seven plots for $1 \leq \lambda \leq 2.2$. The code that generates the plot for $\lambda = \pi/2$ is very similar.

```
function delayedlogistic
%This function solves and plots the solution of the logistic DDE
%                   dy/dx = lambda*y(x)*(1-y(x-1))
%for various values of lambda with intial history s=0.5 on -1\leq x\leq 0
for lambda=1:0.2:2.2
sol = dde23(@ldde,1,@history,[0,30]);
figure;
plot(sol.x,sol.y)
xlabel('Dimensionless time x');
ylabel('y');
title(['\lambda=',num2str(lambda)])
end

function s = history(x)
s=1/2;
end

function dydx= ldde(x,y,z)
ydelay = z;
dydx = lambda*y*(1-ydelay);
end

end
```