MAT 546 Programming Project

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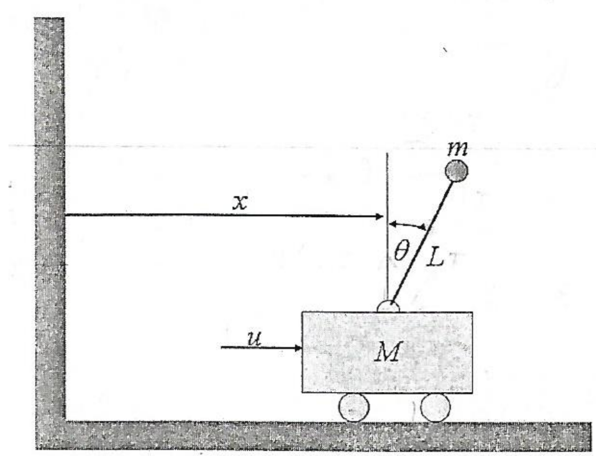
Abstract

The purpose of this paper is to derive the equations of motion of the inverted pendulum problem (see Figure 1) $\dot{X} = AX + bu$ and to find the appropriate control $u = -KX$, which allows the pendulum to remain upright on the interval $0 \leq t \leq 5$. To find this K we use Ackermann's formula with eigenvalues $-2 \pm 2\sqrt{3}i, -10$, and -10. We plot the solutions of our differential equation in Matlab and find that the pendulum indeed remains in upright position at the end of our time interval.

Introduction

Consider an inverted pendulum of mass m attached to a cart through a rod of length L and moment of inertia I. The cart of mass M is being pushed by an external force u, whose purpose is to keep the pendulum upright. Refer to figure 1 to see an illustration of the problem being studied here.

Figure 1: The Inverted Pendulum



The horizontal distance of the pendulum (and, thus, the cart's) from its stability point (where it is upright) is denoted by x, and the angle between the rod and the axis of symmetry of the pendulum is θ. We now derive the equations of motion using the Lagrangian

$$\mathcal{L} = T - V \tag{1}$$

where T is the kinetic energy of the system and V is the potential energy of the system. The motion of the cart and the pendulum, as well as the rotational motion of the rod add kinetic energy to the system, namely

$$T = \frac{1}{2}mv_m^2 + \frac{1}{2}Mv_M^2 + \frac{1}{2}I\omega^2 \tag{2}$$

where v_m is the tangential speed of the pendulum, v_M is the speed of the cart, and ω is the angular speed of the rod. Furthermore, the potential energy of the system is only gravitational, where the hinge that connects the rod and the cart is taken as the zero point, that is

$$V = mgL\cos\theta. \tag{3}$$

Observe that the cart only has horizontal motion $x_M = x$, whereas the pendulum moves along two directions, namely, $x_m = x + L\sin\theta$ and $y_m = L\cos\theta$. Hence, we obtain the following equations

$$\begin{aligned} v_M &= \dot{x}_M = \dot{x} \\ \dot{x}_m &= \dot{x} + L\dot{\theta}\cos\theta \\ \dot{y}_m &= -L\dot{\theta}\sin\theta. \end{aligned}$$

The tangential speed of the pendulum is given by $v_m^2 = \dot{x}_m^2 + \dot{y}_m^2$, so from the equations above we get

$$v_m^2 = (\dot{x} + L\dot{\theta}\cos\theta)^2 + (-L\dot{\theta}\sin\theta)^2 = \dot{x}^2 + 2L\dot{x}\dot{\theta}\cos\theta + L^2\dot{\theta}^2.$$

Using our results above, along with the fact that $\omega = \dot{\theta}$, and substituting them in (2) and (3), and then in (1), we obtain our Lagrangian

$$\mathcal{L} = \frac{1}{2}(m+M)\dot{x}^2 + mL\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}(mL^2 + I)\dot{\theta}^2 - mgL\cos\theta. \tag{4}$$

We will use the Euler-Lagrange equation of the Lagrangian $\mathcal{L}(q_i, \dot{q}_i; t)$, for a coordinate q_i, and t is the independent time variable, namely

$$\left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i} - \frac{\partial}{\partial q_i}\right)\mathcal{L} = D_i \tag{5}$$

where D_i is the dissipation of the coordinate q_i. We consider our coordinates to be $q_1 = x$ and $q_2 = \theta$. Observe that from (4) and considering that u is a non-conservative force, we get

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{x}} &= (m+M)\dot{x} + mL\dot{\theta}\cos\theta \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} &= (m+M)\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta \\ \frac{\partial \mathcal{L}}{\partial x} &= 0 \\ D_1 &= u. \end{aligned}$$

Hence, the Lagrangian method (5) with $q_1 = x$ gives us

$$(m+M)\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta = u. \tag{6}$$

Furthermore, applying (5) to (4) with $q_2 = \theta$ and observing that tension is a restoring force yields

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= mL\dot{x}\cos\theta + (mL^2 + I)\dot{\theta} \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= mL\ddot{x}\cos\theta - mL\dot{x}\dot{\theta}\sin\theta + (mL^2 + I)\ddot{\theta} \\ \frac{\partial \mathcal{L}}{\partial \theta} &= -mL\dot{x}\dot{\theta}\sin\theta + mgL\sin\theta \\ D_2 &= 0. \end{aligned}$$

Hence, we obtain

$$mL\ddot{x}\cos\theta + (mL^2 + I)\ddot{\theta} - mgL\sin\theta = 0. \tag{7}$$

Let us assume θ is small and has slow oscillations, that is, $\cos\theta \approx 1$, $\sin\theta \approx \theta$, and $\dot{\theta}^2 \approx 0$. For instance, equations (6) and (7) reduce to our following equations of motion

$$(m+M)\ddot{x} + mL\ddot{\theta} = u \tag{8}$$

$$mL\ddot{x} + (mL^2 + I)\ddot{\theta} - mgL\theta = 0. \tag{9}$$

Moreover, if we assume the rotational inertia of the pendulum rod is $I = 0$, and after some algebra (8) and (9) become

$$\ddot{\theta} = \frac{m+M}{ML}g\theta - \frac{1}{ML}u \tag{10}$$

$$\ddot{x} = -\frac{mg}{M}\theta + \frac{1}{M}u. \tag{11}$$

Discussion and Results

Now that we have derived our equations of motion, we rewrite the system (10) and (11) using the variables $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x$, and $x_4 = \dot{x}$. Hence

$$\begin{aligned}
\dot{x}_1 &= \dot{\theta} = x_2 \\
\dot{x}_2 &= \ddot{\theta} = \frac{m+M}{ML}gx_1 - \frac{1}{ML}u \\
\dot{x}_3 &= \dot{x} = x_4 \\
\dot{x}_4 &= \ddot{x} = -\frac{mg}{M}x_1 + \frac{1}{M}u.
\end{aligned}$$

Our system now takes the following form

$$\dot{X} = AX + bu \tag{12}$$

where $X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$, $b = \begin{bmatrix} 0 & -\frac{1}{ML} & 0 & \frac{1}{M} \end{bmatrix}^T$, and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{m+M}{ML}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{bmatrix}.$$

For this project, we use $M = 2$, $m = 0.1$, $L = 0.5$, and $g = 9.81$. Also, the initial conditions are in the vector $X(t=0) = \begin{bmatrix} 0.01 & -0.1 & 1 & 2 \end{bmatrix}^T$. If we let $u = 0$, then we have no controlling force, so the pendulum cannot stay upright and falls. Also, by Newton's first law, the cart should move indefinitely. The system $\dot{X} = AX$ is solved with Matlab on the interval $t \in [0, 5]$, and figures 2 and 3 are the solution curves for θ and x, respectively.

Figure 2: Angle with the vertical $\theta(t)$ with $u = 0$.

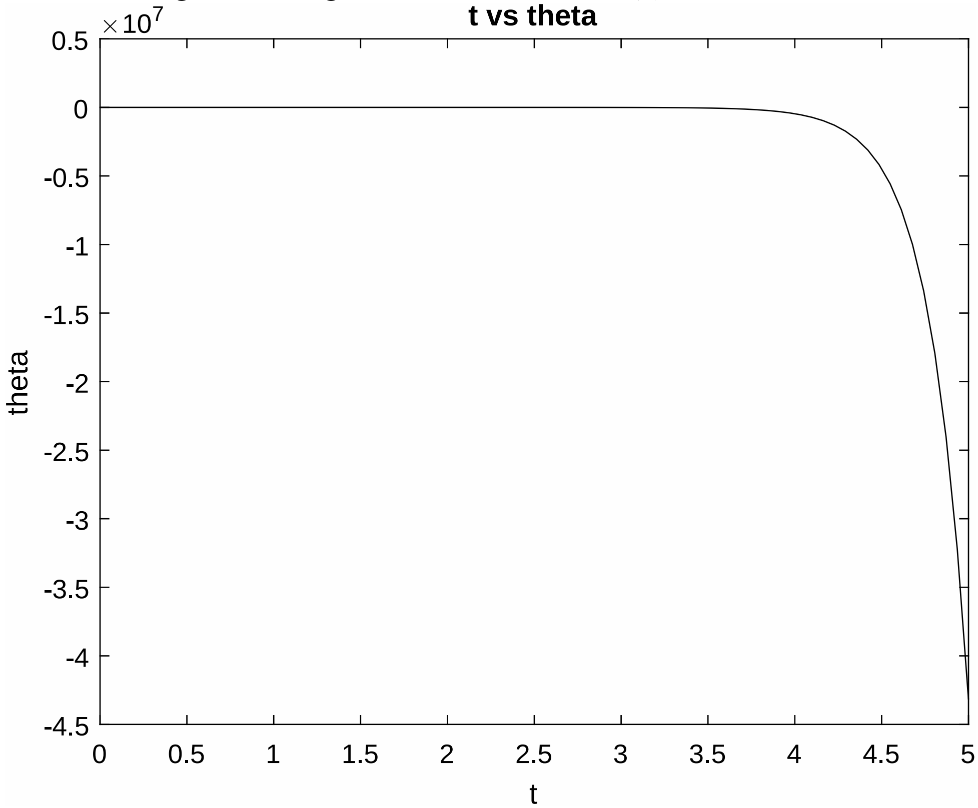


Figure 3: Horizontal Position $x(t)$ with $u = 0$.

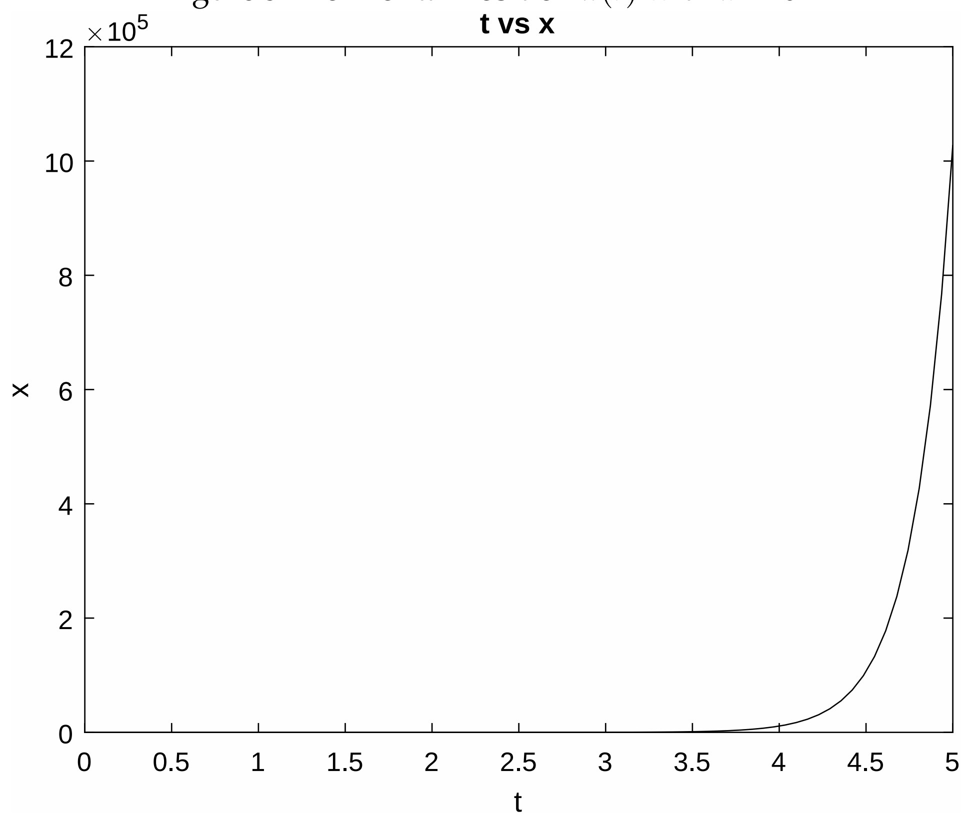


Figure 2 suggests that our pendulum goes in circles clockwise indefinitely (assuming the surface of the cart does not stop it), and figure 3 suggests the cart moves in the $+x$ direction indefinitely. Hence, we want to compute a row vector K such that $u = -KX$, which imposes to the system in (12) the eigenvalues $\lambda = -2 \pm 2\sqrt{3}i$ each of multiplicity one, and $\lambda = -10$ of multiplicity two. Note that these eigenvalues are all on the left half of the complex plane. To obtain K, we use Ackermann's formula

$$K = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} \varphi(A) \tag{13}$$

where $\varphi(\lambda)$ is the characteristic polynomial of the system after pole-placement, namely

$$\varphi(\lambda) = (\lambda + 2 + 2\sqrt{3}i)(\lambda + 2 - 2\sqrt{3}i)(\lambda + 10)^2 = \lambda^4 + 24\lambda^3 + 196\lambda^2 + 720\lambda + 1600.$$

Using our formula in (13) in Matlab gives the following value of K

$$K = \begin{bmatrix} -298.1504 & -60.6972 & -163.0989 & -73.3945 \end{bmatrix}.$$

When we solve our system in (12) with $u = -KX$, we get the plots in figures 4 and 5.

Figure 4: Angle with the vertical $\theta(t)$ with $u = -KX$.

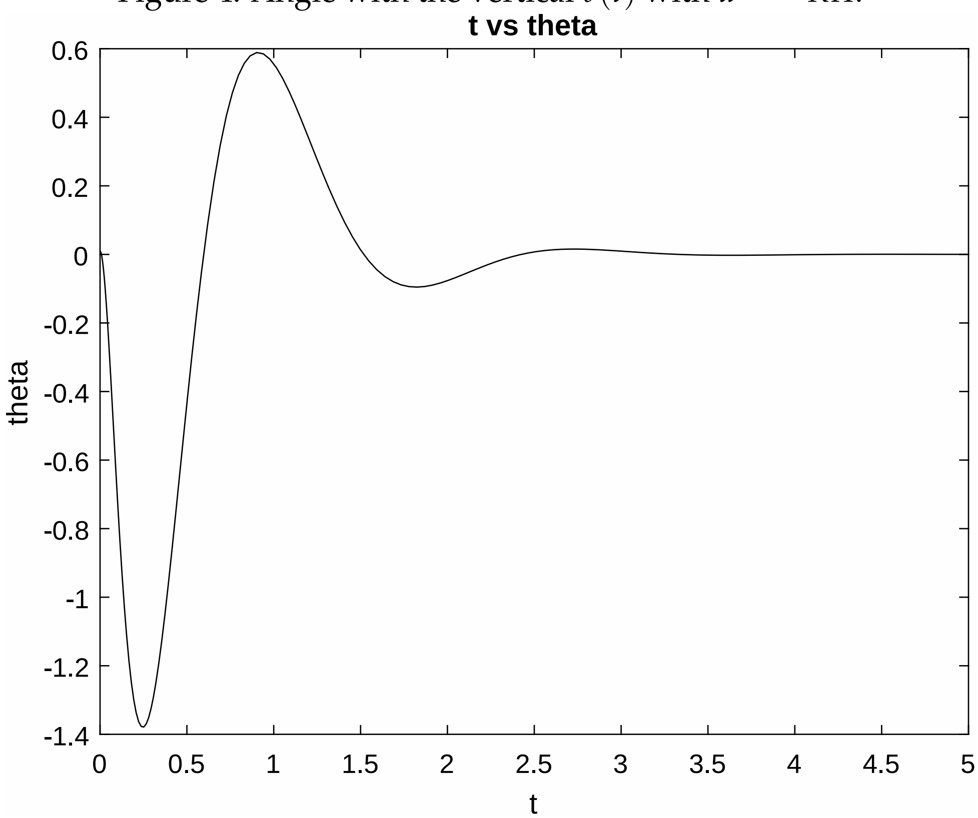
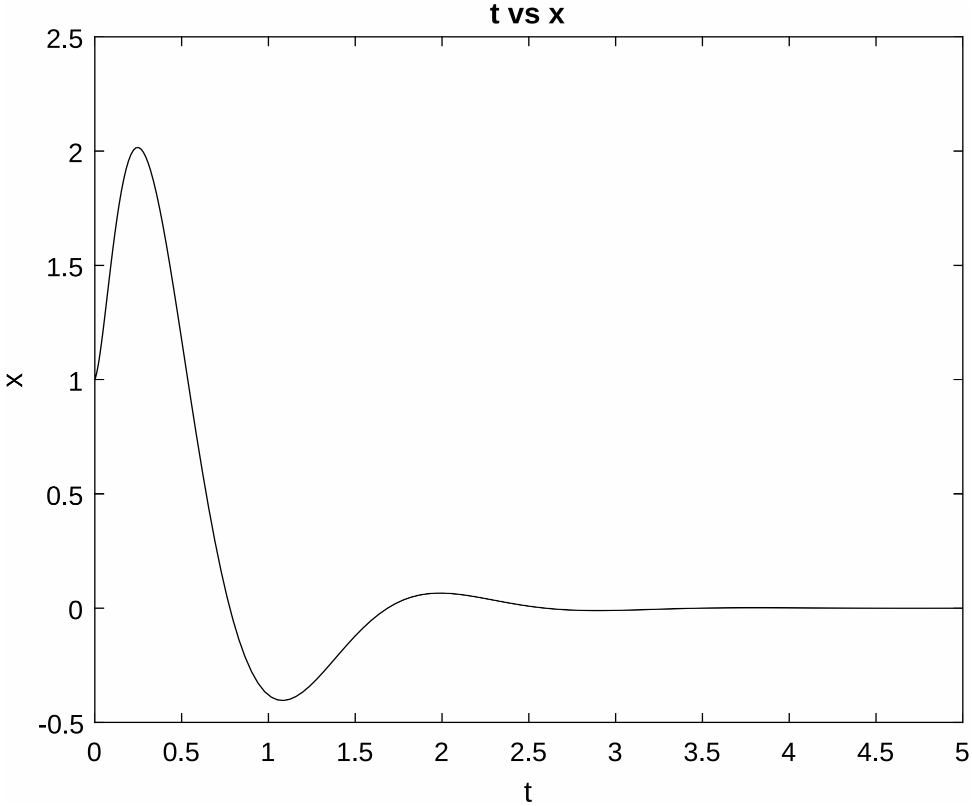


Figure 5: Horizontal Position $x(t)$ with $u = -KX$.



Observe that the control force worked because figure 4 shows that θ eventually becomes stable making an angle of zero with the vertical. Furthermore, figure 5 shows that the cart eventually becomes stationary, i.e. $x(t)$ stays very close to zero, so the pendulum remains upright.

References

[1] ControlTheoryPro.com. Lagrange Equations of Motion for NonConservative Forces.

[2] David Morin. Introduction to Classical Mechanics with Problems and Solutions. Cambridge Editorial.

[3] Wikipedia. Lagrangian Mechanics.

Source Codes

This is the Matlab code which solves and plots the system in (12) for both cases, that is, $u = 0$ and $u = -KX$.

```
tspan = [0, 5];
in_val = [0.01, -0.1 ,1 ,2 ];

[t, X] = ode45(@pendulum, tspan, in_val);
x1 = X(:,1);
x3 = X(:,3);
figure(1)
plot(t,x1); xlabel('t'); ylabel('theta'); title('t vs theta');
figure(2)
plot(t,x3); xlabel('t'); ylabel('x'); title('t vs x');

function Xdot = pendulum(t,X)
M=2; m=0.1; L=0.5; g=9.81;
A = [ 0 1 0 0; (M+m)*g/(M*L) 0 0 0; 0 0 0 1; -m*g/M 0 0 0 ];
b = [0; -1/(M*L); 0; 1/M];
%
phi = A^4+24*A^3+196*A^2+720*A+1600*eye(4);
K = [0 0 0 1]*inv([b, A*b, A^2*b, A^3*b])*phi;
%
Xdot = A*X -(K*X)*b; %this line is just Xdot=A*X if u=0
end
```